Upskilling Proof and argument

Examples

Based on P1/P2

1. Show that the equation $2x^3 + x^2 + 8x + 4 = 0$ has only 1 real root.

2. C is the curve with equation $y = x^3 - x^2 - 8x + 12$

   Show that the *x* axis is a tangent to C.

3. Given that *x* and *y* are positive, show that $\frac{x}{y} + \frac{y}{x} \geq 2$
4. Show that the sum of the first *n* terms of the series

   $1 + \frac{3}{4} + \frac{9}{16} + \cdots$.. can never be greater than 4

5. Show that the sum of any 3 successive terms of an Arithmetic Series is a multiple of 3.

6. The terms of a Geometric Series are all positive whole numbers. Show that the product of any 3 successive terms is a perfect cube.

7. Prove by exhaustion that no square number leaves a remainder of 2 when divided by 5

8. $\mathbf{C_1}$ is the curve with equation $y = 27x(4 - x)$. $\mathbf{C_2}$ is the curve with equation $y = \frac{256}{x}$.

   Show that the curve C_1 touches the curve $\mathbf{C_2}$.

9. Let *N* be a 4 digit number. Let *m* be the sum of the digits of *N.*
   Prove that *N* − *m* is a multiple of 9

10. Generalise the result in Q9.
    What can you conclude about the well known divisibility by 9 test?

Answers

1. $2x^3 + x^2 + 8x + 4 = (2x + 1)(x^2 + 4)$ so the only real root is $x = -\frac{1}{2}$
2. $x^3 - x^2 - 8x + 12 = (x - 2)^2\ (x + 3)$ so roots of $x^3 - x^2 - 8x + 12 = 0$ are at x = 2 and x = –3.
   The root at x = 2 is repeated so the x-axis is a tangent to the curve (at x = 2).
   (Alternatively $y' = 3x^2 - 2x - 8 = 0$ at x = 2)
3. Let $t = \frac{x}{y}$ then $\frac{x}{y} + \frac{y}{x} = t + \frac{1}{t}$ F(t) $= t + \frac{1}{t}$ $\frac{dF}{dt} = 1 - \frac{1}{t^2} = 0$ when t = 1 and $\frac{d^2F}{dt^2} = \frac{2}{t^3} > 0$
   So a minimum when t = 1 so x = y and the minimum value of $\frac{x}{y} + \frac{y}{x}$ is 2

4. The series is always increasing, so look at the sum to infinity, S. $S = \frac{1}{1-\frac{3}{4}} = 4$
5. $a + (n-1)d + a + nd + a + (n+1)d$ = $3a + 3nd$ = $3(a + nd)$ Hence result.

6. $ar^n \times ar^{n+1} \times ar^{n+2}$ = a^3r^{3n+3} = $(ar^{n+1})^3$. Hence result.

7. Method 1 The first 10 squares are 1, 4, 9, (1)6, (2)5, (3)6, (4)9, (6)4 (8)1 10(0)
   So the units digit of any squares are 1, 4. 9. 6, 5, 6, 9, 4, 1, 0
   When dividing <u>any</u> square by 5 the remainder is will be one of 1, 4. 9. 6, 5, 6, 9, 4, 1, 0
   Hence result.
   Method 2. Any whole number can be written as $5k$, $5k \pm 1$, $5k \pm 2$
   Squaring these gives $25k^2$ or $25k^2 \pm 10k + 1$ or $25k^2 \pm 20k + 4$ which produces remainders 0, 1 and 4 respectively.
   Hence result.

8. $27x(4 - x) = \frac{256}{x}$ So $27x^3 - 108x^2 + 256 = 0$ is the condition the curves meet.
   $108 - 54x = -\frac{256}{x^2}$ So $54x^3 - 108x^2 - 256 = 0$ is the condition for equal gradients.
   The equations are consistent when $x = \frac{8}{3}$ so $\mathbf{C_1}$ is a tangent to $\mathbf{C_2}$

9. Set N = 1000x + 100y + 10z + w (all positive integers) Then m = x + y + z + w

   N - m = 1000x + 100y + 10z + w − (x + y + z + w) = 999x + 99y + 9z = 9(111x + 11y + z)

   Hence result

10. In the general case, consider $10^n a - a = a\ (10^n - 1)$.
    $10^n - 1$ = 9999…99 so (n −1) 9s and so is divisible by 9

For the divisibility test -
Since N - m is a multiple of 9, then if m is a multiple of 9 N must also be.